

THIRD HOMOLOGY OF GENERAL LINEAR GROUPS OVER RINGS WITH MANY UNITS

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ABSTRACT. For a commutative ring R with many units, we describe the kernel of $H_3(\text{inc}) : H_3(\text{GL}_2(R), \mathbb{Z}) \rightarrow H_3(\text{GL}_3(R), \mathbb{Z})$. Moreover we show that the elements of this kernel are of order at most two. As an application we study the indecomposable part of $K_3(R)$.

INTRODUCTION

Interest in the study of the homology of general linear groups has arose mostly because of their close connection with the K -theory of rings. For any ring R and any positive integer n , there are natural homomorphisms

$$\begin{array}{ccc} K_n(R) & \xrightarrow{h_n} & H_n(\text{GL}(R), \mathbb{Z}) \\ h'_n \searrow & & \swarrow \nearrow \\ & H_n(E(R), \mathbb{Z}), & \end{array}$$

where $E(R)$ is the elementary subgroup of the stable general linear group $\text{GL}(R)$ and h_n and h'_n ($n \geq 2$ for h'_n) are the Hurewicz maps coming from algebraic topology [10, Chap. 2].

It is known that $K_1(R) \xrightarrow{h_1} H_1(\text{GL}(R), \mathbb{Z})$, $K_2(R) \xrightarrow{h'_2} H_2(E(R), \mathbb{Z})$ [10, Chap. 2]. The homomorphism $h'_3 : K_3(R) \rightarrow H_3(E(R), \mathbb{Z})$ is surjective with 2-torsion kernel [12, Corollary 5.2], [9, Proposition 2.5].

Homological stability type theorems, are very powerful tools for the study of K -theory of rings. Suslin has proved that for an infinite field F , we have the homological stability

$$H_n(\text{GL}_n(F), \mathbb{Z}) \xrightarrow{\sim} H_n(\text{GL}_{n+1}(F), \mathbb{Z}) \xrightarrow{\sim} H_n(\text{GL}_{n+1}(F), \mathbb{Z}) \xrightarrow{\sim} \cdots,$$

and used this to prove many interesting results [11]. For example he showed that we have an exact sequence

$$H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \xrightarrow{H_n(\text{inc})} H_n(\text{GL}_n(F), \mathbb{Z}) \longrightarrow K_n^M(F) \longrightarrow 0.$$

Suslin has conjectured that the kernel of

$$H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \rightarrow H_n(\text{GL}_n(F), \mathbb{Z})$$

is a torsion group [9, Problem 4.13]. These results can be generalized over rings with many units [4], e.g. semilocal rings with infinite residue fields. Also Suslin's conjecture can be asked in this more general setting [7]. A positive answer to this conjecture only is known for $n \leq 4$ [3], [8], [7].

It was known that when F is an infinite field, the kernel of the homomorphism $H_3(\mathrm{GL}_2(F), \mathbb{Z}) \rightarrow H_3(\mathrm{GL}_3(F), \mathbb{Z})$ is a 2-power torsion group [8]. In this article we generalize this to all commutative rings with many units. In fact we do more. Here we describe the kernel of

$$H_3(\mathrm{inc}) : H_3(\mathrm{GL}_2(R), \mathbb{Z}) \rightarrow H_3(\mathrm{GL}_3(R), \mathbb{Z}),$$

where R is a commutative ring with many units. Our main theorem claims that the elements of $\ker(H_3(\mathrm{inc}))$ are of the form

$$\sum \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(c, c^{-1}))$$

provided that

$$\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in R^* \otimes_{\mathbb{Z}} K_2^M(R).$$

Moreover by an easy argument we will show that $\ker(H_3(\mathrm{inc}))$ is a 2-torsion group. It is highly expected that this kernel should be trivial, at least when R is a field [5, Section 5].

It is known that, the map $H_3(\mathrm{inc})$ is closely related to the indecomposable part of $K_3(R)$, i.e. $K_3(R)^{\mathrm{ind}} := K_3(R)/K_3^M(R)$ [8], [5]. As an application of our main theorem we show that

$$K_3(R)^{\mathrm{ind}} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \simeq H_0(R^*, H_3(\mathrm{SL}_2(R), \mathbb{Z}[1/2])).$$

If $R^* = R^{*2}$, then we get the isomorphism

$$K_3(R)^{\mathrm{ind}} \simeq H_3(\mathrm{SL}_2(R), \mathbb{Z}).$$

Previously these results were only known for infinite fields [8].

Notation. In this article by $H_i(G)$ we mean the homology of group G with integral coefficients, namely $H_i(G, \mathbb{Z})$. By GL_n (resp. SL_n) we mean the general (resp. special) linear group $\mathrm{GL}_n(R)$ (resp. $\mathrm{SL}_n(R)$), where R is a commutative ring with 1. If $A \rightarrow A'$ is a homomorphism of abelian groups, by A'/A we mean $\mathrm{coker}(A \rightarrow A')$ and we take other liberties of this kind. For a group A , by $A_{\mathbb{Z}[1/2]}$ we mean $A \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$.

1. THIRD HOMOLOGY OF PRODUCT OF TWO ABELIAN GROUPS

In this section we will study the homology group $H_3(A \times B)$, where A and B are abelian groups.

First we assume $A = B = \mathbb{Z}/n$. By applying the Künneth formula [13, Proposition 6.1.13] to $H_3(\mathbb{Z}/n \times \mathbb{Z}/n)$ and using the calculation

of the homology of finite cyclic groups [13, Theorem 6.2.2, Example 6.2.3], we obtain the exact sequence

$$0 \rightarrow H_3(\mathbb{Z}/n) \oplus H_3(\mathbb{Z}/n) \rightarrow H_3(\mathbb{Z}/n \times \mathbb{Z}/n) \rightarrow \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/n) \rightarrow 0.$$

If $p_i : \mathbb{Z}/n \times \mathbb{Z}/n \rightarrow \mathbb{Z}/n$, $i = 1, 2$, is projection on the i -th factor, then

$$(p_{1*}, p_{2*}) : H_3(\mathbb{Z}/n \times \mathbb{Z}/n) \rightarrow H_3(\mathbb{Z}/n) \oplus H_3(\mathbb{Z}/n)$$

splits the above exact sequence. Thus we obtain a canonical splitting map

$$\theta_{n,n} : \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/n) \rightarrow H_3(\mathbb{Z}/n \times \mathbb{Z}/n).$$

If $\langle \bar{1}, n, \bar{1} \rangle$ is the image of $\bar{1} \in \mathbb{Z}/n$ under the isomorphism

$$\mathbb{Z}/n \xrightarrow{\sim} \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/n),$$

then one can show that $\theta_{n,n}(\langle \bar{1}, n, \bar{1} \rangle) = \chi_{n,n}$, where

$$\begin{aligned} \chi_{n,n} := \sum_{i=1}^n & ([(\bar{1}, 0)|(0, \bar{1})|(0, \bar{i})] - [(0, \bar{1})|(\bar{1}, 0)|(0, \bar{i})] \\ & + [(0, \bar{1})|(0, \bar{i})|(\bar{1}, 0)] + [(\bar{1}, 0)|(\bar{i}, 0)|(0, \bar{1})] \\ & - [(\bar{1}, 0)|(0, \bar{1})|(\bar{i}, 0)] + [(0, \bar{1})|(\bar{1}, 0)|(\bar{i}, 0)]). \end{aligned}$$

[6, Chap. V, Proposition 10.6], [8, Proposition 4.1]. If $A = \mathbb{Z}/m$ and $B = \mathbb{Z}/n$, then the same approach shows that the exact sequence

$$0 \rightarrow H_3(\mathbb{Z}/m) \oplus H_3(\mathbb{Z}/n) \rightarrow H_3(\mathbb{Z}/m \times \mathbb{Z}/n) \rightarrow \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) \rightarrow 0,$$

splits canonically. The splitting map

$$\theta_{m,n} : \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) \rightarrow H_3(\mathbb{Z}/m \times \mathbb{Z}/n)$$

can be computed similar to $\theta_{n,n}$. In fact if $\langle \overline{m/d}, d, \overline{n/d} \rangle$ is the image of $\bar{1} \in \mathbb{Z}/(m, n)$ under the isomorphism $\mathbb{Z}/(m, n) \simeq \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n)$, then $\theta_{m,n}(\langle \overline{m/d}, d, \overline{n/d} \rangle) = \chi_{m,n}$, where

$$\begin{aligned} \chi_{m,n} := \sum_{i=1}^n & ([(\frac{\overline{m}}{d}, 0)|(0, \frac{\overline{n}}{d})|(0, \frac{\overline{in}}{d})] - [(0, \frac{\overline{n}}{d})|(\frac{\overline{m}}{d}, 0)|(0, \frac{\overline{in}}{d})] \\ & + [(0, \frac{\overline{n}}{d})|(0, \frac{\overline{in}}{d})|(\frac{\overline{m}}{d}, 0)] + [(\frac{\overline{m}}{d}, 0)|(\frac{\overline{im}}{d}, 0)|(0, \frac{\overline{n}}{d})] \\ & - [(\frac{\overline{m}}{d}, 0)|(0, \frac{\overline{n}}{d})|(\frac{\overline{im}}{d}, 0)] + [(0, \frac{\overline{n}}{d})|(\frac{\overline{m}}{d}, 0)|(\frac{\overline{im}}{d}, 0)]). \end{aligned}$$

In the next proposition we extend these results to all abelian groups.

Proposition 1.1. *Let A and B be abelian groups. Then we have the canonical decomposition*

$$H_3(A \times B) = \bigoplus_{i+j=3} H_i(A) \otimes H_j(B) \oplus \text{Tor}_1^{\mathbb{Z}}(A, B).$$

Proof. By the Künneth formula we have the exact sequence

$$0 \longrightarrow \bigoplus_{i+j=3} H_i(A) \otimes H_j(B) \longrightarrow H_3(A \times B) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A, B) \longrightarrow 0.$$

We will construct a canonical splitting map

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \longrightarrow H_3(A \times B).$$

It is known that direct limit with directed set index, is an exact functor and it commutes with the homology group [2, Chap. V, Section 5, Exercise 3] and the functor Tor [13, Corollary 2.6.17]. Since any abelian group can be written as direct limit of its finitely generated subgroups, we may assume that A and B are finitely generated abelian groups. On the other hand,

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(A_{\mathrm{tor}}, B_{\mathrm{tor}}),$$

where A_{tor} is the subgroup of torsion elements of A . So we may even assume that A and B are finite abelian groups. Let

$$A = \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_r, \quad B = \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_s.$$

Now consider the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow H_3(\mathbb{Z}/m_i) \oplus H_3(\mathbb{Z}/n_j) & \rightarrow & H_3(\mathbb{Z}/m_i \times \mathbb{Z}/n_j) & \rightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m_i, \mathbb{Z}/n_j) & \rightarrow & 0 \\ & & \downarrow & & \downarrow \mathrm{inc}_{m_i, n_j} & & \downarrow \mathrm{inc} \\ 0 \rightarrow \bigoplus_{i+j=3} H_i(A) \otimes H_j(B) & \rightarrow & H_3(A \times B) & \rightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(A, B) & \rightarrow & 0. \end{array}$$

We have seen that the first row of this diagram splits by the canonical map θ_{m_i, n_j} . Thus the composition

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m_i, \mathbb{Z}/n_j) \xrightarrow{\mathrm{inc}_{m_i, n_j} \circ \theta_{m_i, n_j}} H_3(A \times B) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A, B)$$

is the natural inclusion map. Since

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B) = \bigoplus_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m_i, \mathbb{Z}/n_j),$$

we obtain a map $\theta_{A, B} : \mathrm{Tor}_1^{\mathbb{Z}}(A, B) \longrightarrow H_3(A \times B)$ that decomposes our exact sequence canonically. In fact $\theta_{A, B} = \sum_{i, j} \mathrm{inc}_{m_i, n_j} \circ \theta_{m_i, n_j}$. \square

2. THE THIRD HOMOLOGY OF GL_2

A commutative ring R with 1 is called a *ring with many units* if for any $n \geq 2$ and for any finite number of surjective linear forms $f_i : R^n \rightarrow R$, there exists a $v \in R^n$ such that, for all i , $f_i(v) \in R^*$. Important examples of rings with many units are semilocal rings with infinite residue fields. In particular for an infinite field F , any commutative

finite dimensional F -algebra is a semilocal ring and so is a ring with many units. In this article we always assume that R is a commutative ring with many units.

Let

$$R^{*3} \times \mathrm{GL}_0 \xrightarrow{\mathrm{inc}} R^{*2} \times \mathrm{GL}_1 \xrightarrow{\mathrm{inc}} R^* \times \mathrm{GL}_2 \xrightarrow{\mathrm{inc}} \mathrm{GL}_3$$

be the natural diagonal inclusions. Here by R^{*n} we mean $R^* \times \cdots \times R^*$ (n -times). Let

$$\sigma_2^1 := \mathrm{inc} : R^* \times \mathrm{GL}_2 \longrightarrow \mathrm{GL}_3,$$

$$\sigma_1^1 : R^{*2} \times \mathrm{GL}_1 \longrightarrow R^* \times \mathrm{GL}_2, \quad (a, b, c) \mapsto (b, a, c),$$

$$\sigma_1^2 = \mathrm{inc} : R^{*2} \times \mathrm{GL}_1 \longrightarrow R^* \times \mathrm{GL}_2, \quad (a, b, c) \mapsto (a, b, c),$$

$$\sigma_0^1 : R^{*3} \times \mathrm{GL}_0 \longrightarrow R^{*2} \times \mathrm{GL}_1, \quad (a, b, c) \mapsto (b, c, a),$$

$$\sigma_0^2 : R^{*3} \times \mathrm{GL}_0 \longrightarrow R^{*2} \times \mathrm{GL}_1, \quad (a, b, c) \mapsto (a, c, b),$$

$$\sigma_0^3 = \mathrm{inc} : R^{*3} \times \mathrm{GL}_0 \longrightarrow R^{*2} \times \mathrm{GL}_1, \quad (a, b, c) \mapsto (a, b, c).$$

It is easy to see that the chain of maps

$$\begin{aligned} H_3(R^{*3} \times \mathrm{GL}_0) &\xrightarrow{\sigma_0^1 - \sigma_0^2 + \sigma_0^3} H_3(R^{*2} \times \mathrm{GL}_1) \\ &\xrightarrow{\sigma_1^1 - \sigma_1^2} H_3(R^* \times \mathrm{GL}_2) \xrightarrow{\sigma_2^1} H_3(\mathrm{GL}_3) \longrightarrow 0 \end{aligned}$$

is a chain complex. The following result has been proved in [8, Corollary 3.5].

Theorem 2.1. *The sequence*

$$H_3(R^{*2} \times \mathrm{GL}_1) \xrightarrow{\sigma_1^1 - \sigma_1^2} H_3(R^* \times \mathrm{GL}_2) \xrightarrow{\sigma_2^1} H_3(\mathrm{GL}_3) \longrightarrow 0$$

is exact.

Using the Künneth formula [13, Proposition 6.1.13], we have the decomposition $H_3(R^* \times \mathrm{GL}_2) = \bigoplus_{i=0}^4 S_i$, where

$$S_0 = H_3(\mathrm{GL}_2),$$

$$S_i = H_i(R^*) \otimes H_{3-i}(\mathrm{GL}_2), \quad 1 \leq i \leq 3,$$

$$S_4 = \mathrm{Tor}_1^{\mathbb{Z}}(R^*, H_1(\mathrm{GL}_2)) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mu(R), \mu(R)).$$

Note that by the homological stability, $R^* \simeq H_1(\mathrm{GL}_1) \simeq H_1(\mathrm{GL}_2)$ [4, Theorem 1]. This decomposition is canonical. The splitting map

$$S_4 \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mu(R), \mu(R)) \longrightarrow H_3(R^* \times \mathrm{GL}_2)$$

is given by the composition

$$S_4 \simeq \operatorname{Tor}_1^{\mathbb{Z}}(\mu(R), \mu(R)) \xrightarrow{\theta_{R,R}} H_3(R^* \times R^*) \xrightarrow{q^*} H_3(R^* \times \operatorname{GL}_2),$$

where

$$q : R^* \times R^* \longrightarrow R^* \times \operatorname{GL}_2, \quad (a, b) \mapsto (a, b, 1),$$

and $\theta_{R,R}$ is obtained from Proposition 1.1. Using the decomposition

$$H_2(\operatorname{GL}_2) = H_2(\operatorname{GL}_1) \oplus K_2^M(R)$$

[4, Theorem 2], we have $S_1 = S'_1 \oplus S''_1$, where

$$S'_1 = R^* \otimes H_2(\operatorname{GL}_1), \quad S''_1 = R^* \otimes K_2^M(R).$$

We should remark that the inclusion $K_2^M(R) \longrightarrow H_2(\operatorname{GL}_2)$, in the decomposition of $H_2(\operatorname{GL}_2)$, is given by the formula

$$\{a, b\} \mapsto \mathbf{c}(\operatorname{diag}(a, 1), \operatorname{diag}(b, b^{-1}))$$

[3, Proposition A.11]. For the definition of Milnor's K -groups, $K_n^M(R)$, over commutative rings and their study over rings with many units, we refer the interested readers to subsection 3.2 of [4].

Let us to introduce the notation $\mathbf{c}(-, -)$ in a more general setting and state some of its main properties. These will be used frequently in this article. Let G be a group and set

$$\mathbf{c}(g_1, g_2, \dots, g_n) := \sum_{\sigma \in \Sigma_n} \operatorname{sign}(\sigma) [g_{\sigma(1)} | g_{\sigma(2)} | \dots | g_{\sigma(n)}] \in H_n(G),$$

where $g_1, \dots, g_n \in G$ pairwise commute and Σ_n is the symmetric group of degree n . Here we use the bar resolution of G [2, Chapter I, Section 5] to define the homology of G .

Lemma 2.2. *Let G and G' be two groups.*

(i) *If $h_1 \in G$ commutes with all the elements $g_1, \dots, g_n \in G$, then*

$$\mathbf{c}(g_1 h_1, g_2, \dots, g_n) = \mathbf{c}(g_1, g_2, \dots, g_n) + \mathbf{c}(h_1, g_2, \dots, g_n).$$

(ii) *For every $\sigma \in \Sigma_n$, $\mathbf{c}(g_{\sigma(1)}, \dots, g_{\sigma(n)}) = \operatorname{sign}(\sigma) \mathbf{c}(g_1, \dots, g_n)$.*

(iii) *The cup product of $\mathbf{c}(g_1, \dots, g_p) \in H_p(G)$ and $\mathbf{c}(g'_1, \dots, g'_q) \in H_q(G')$ is $\mathbf{c}((g_1, 1), \dots, (g_p, 1), (1, g'_1), \dots, (1, g'_q)) \in H_{p+q}(G \times G')$.*

Proof. The proofs follow from direct computations, so we leave it to the interested readers. \square

Again using the Künneth formula and Proposition 1.1, we obtain the canonical decomposition $H_3(R^{*2} \times \operatorname{GL}_1) = \bigoplus_{i=0}^8 T_i$, where

$$\begin{aligned}
T_0 &= H_3(\mathrm{GL}_1), \\
T_1 &= \bigoplus_{i=1}^3 T_{1,i} = \bigoplus_{i=1}^3 H_i(R_1^*) \otimes H_{3-i}(\mathrm{GL}_1), \\
T_2 &= \bigoplus_{i=1}^3 T_{2,i} = \bigoplus_{i=1}^3 H_i(R_2^*) \otimes H_{3-i}(\mathrm{GL}_1), \\
T_3 &= R_1^* \otimes R_2^* \otimes H_1(\mathrm{GL}_1), \\
T_4 &= \mathrm{Tor}_1^{\mathbb{Z}}(R_1^*, R_2^*) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mu(R), \mu(R)), \\
T_5 &= \mathrm{Tor}_1^{\mathbb{Z}}(R_1^*, H_1(\mathrm{GL}_1)) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mu(R), \mu(R)), \\
T_6 &= \mathrm{Tor}_1^{\mathbb{Z}}(R_2^*, H_1(\mathrm{GL}_1)) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mu(R), \mu(R)), \\
T_7 &= R_1^* \otimes H_2(R_2^*), \\
T_8 &= H_2(R_1^*) \otimes R_2^*.
\end{aligned}$$

Here by R_i^* we mean the i -th component of $R^* \times \cdots \times R^*$. Now we give an explicit description of restriction of the map $\alpha := \sigma_{1*}^1 - \sigma_{1*}^2$ on all T_i 's. By direct computations one sees that

$$\begin{aligned}
\alpha|_{T_0} : T_0 &\longrightarrow S_0, & x &\mapsto 0, \\
\alpha|_{T_{1,i}} : T_{1,i} &\longrightarrow S_0 \oplus S_i, & x_i \otimes x'_i &\mapsto (x_i \cup x'_i, -x_i \otimes x'_i), 1 \leq i \leq 3, \\
\alpha|_{T_{2,i}} : T_{2,i} &\longrightarrow S_0 \oplus S_i, & y_i \otimes y'_i &\mapsto (-y_i \cup y'_i, y_i \otimes y'_i), 1 \leq i \leq 3, \\
\alpha|_{T_3} : T_3 &\longrightarrow S_1, & a \otimes b \otimes c &\mapsto -b \otimes (a \cup c) - a \otimes (b \cup c), \\
\alpha|_{T_4} : T_4 &\longrightarrow S_4, & z &\mapsto 0, \\
\alpha|_{T_5} : T_5 &\longrightarrow S_0 \oplus S_4, & u &\mapsto (\sigma_{1*}^1(u), -u), \\
\alpha|_{T_6} : T_6 &\longrightarrow S_0 \oplus S_4, & v &\mapsto (-\sigma_{1*}^2(v), v), \\
\alpha|_{T_7} : T_7 &\longrightarrow S_1 \oplus S_2, & d \otimes u' &\mapsto (-d \otimes u', u' \otimes d), \\
\alpha|_{T_8} : T_8 &\longrightarrow S_1 \oplus S_2, & v' \otimes e &\mapsto (e \otimes v', -v' \otimes e),
\end{aligned}$$

where $x \cup y$ is the cup product of x and y .

3. THE KERNEL OF $H_3(\mathrm{GL}_2) \rightarrow H_3(\mathrm{GL}_3)$

Our goal in this article is to study the kernel of the map $\mathrm{inc}_* : H_3(\mathrm{GL}_2) \rightarrow H_3(\mathrm{GL}_3)$. So let $x \in \ker(\mathrm{inc}_*)$. Then

$$(x, 0, 0, 0, 0) \in \ker(\sigma_{2*}^1) \subseteq \bigoplus_{i=0}^4 S_i = H_3(R^* \times \mathrm{GL}_2).$$

By Theorem 2.1 and by the explicit description of $\alpha = \sigma_{1*}^1 - \sigma_{1*}^2$ given in the previous section, there exists an element

$$l = (0, (x_i \otimes x'_i)_{1 \leq i \leq 3}, (y_i \otimes y'_i)_{1 \leq i \leq 3}, \sum a \otimes b \otimes c, 0, u, v, d \otimes u', v' \otimes e)$$

in $H_3(R^{*2} \times \mathrm{GL}_1)$ such that $\alpha(l) = (x, 0, 0, 0, 0)$.

Set $\beta := \sigma_{0*}^1 - \sigma_{0*}^2 + \sigma_{0*}^3$, and consider the following summands of $H_3(R^{*3} \times \mathrm{GL}_0)$,

$$T'_1 := R_1^* \otimes H_2(R_2^*), \quad T'_2 := H_2(R_1^*) \otimes R_2^*.$$

By easy computations one sees that

$$\begin{aligned} \beta|_{T'_1} : T'_1 &\longrightarrow T_{1,1} \oplus T_{1,2} \oplus T_7, \quad f \otimes w \mapsto (-f \otimes w, w \otimes f, f \otimes w) \\ \beta|_{T'_2} : T'_2 &\longrightarrow T_{1,1} \oplus T_{1,2} \oplus T_8, \quad w' \otimes f' \mapsto (f' \otimes w', -w' \otimes f', w' \otimes f'). \end{aligned}$$

So we may assume $d \otimes u' = 0, v' \otimes e = 0$. Therefore we have

$$\begin{aligned} \sum_{i=1}^3 x_i \cup x'_i - \sum_{i=1}^3 y_i \cup y'_i + \sigma_{1*}^1(u) - \sigma_{1*}^2(v) &= x, \\ -x_1 \otimes x'_1 + y_1 \otimes y'_1 - \sum [b \otimes (a \cup c) + a \otimes (b \cup c)] &= 0, \\ -x_2 \otimes x'_2 + y_2 \otimes y'_2 &= 0, \\ -x_3 \otimes x'_3 + y_3 \otimes y'_3 &= 0, \\ -u + v &= 0. \end{aligned}$$

Therefore we obtain the following relations

$$\begin{aligned} x &= x_1 \cup x'_1 - y_1 \cup y'_1 \in S_0 = H_3(\mathrm{GL}_2), \\ x_1 \otimes x'_1 - y_1 \otimes y'_1 &= -\sum b \otimes (a \cup c) + a \otimes (b \cup c) \in S_1. \end{aligned}$$

Under the decomposition $H_2(\mathrm{GL}_2) = H_2(\mathrm{GL}_1) \oplus K_2^M(R)$, we have

$$a \cup b = \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b)) = (\mathbf{c}(a, b), \{a, b\}).$$

Thus under the decomposition $S_1 = S'_1 \oplus S''_1$, we have

$$(x_1 \otimes x'_1 - y_1 \otimes y'_1 + \sum b \otimes \mathbf{c}(a, c) + a \otimes \mathbf{c}(b, c), \sum b \otimes \{a, c\} + a \otimes \{b, c\}) = 0,$$

and hence

$$\begin{aligned} x_1 \otimes x'_1 - y_1 \otimes y'_1 &= -\sum b \otimes \mathbf{c}(a, c) + a \otimes \mathbf{c}(b, c), \\ \sum b \otimes \{a, c\} + a \otimes \{b, c\} &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} x &= -\sum \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(1, c)) \\ &\quad + \mathbf{c}(\mathrm{diag}(b, 1), \mathrm{diag}(1, a), \mathrm{diag}(1, c)) \\ &= \sum \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(c, c^{-1})), \end{aligned}$$

such that $\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0$. From now on, we will use the following notation:

$$l_{a,b,c} = \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(c, c^{-1}))$$

Hence we have proved most parts of the following theorem.

Theorem 3.1. *Let R be a commutative ring with many units. Then the kernel of $\text{inc}_* : H_3(\text{GL}_2) \rightarrow H_3(\text{GL}_3)$ consists of elements of the form $\sum \mathbf{c}(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(c, c^{-1}))$ provided that*

$$\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in R^* \otimes K_2^M(R).$$

In particular $\ker(\text{inc}_) \subseteq R^* \cup H_2(\text{GL}_1) \subseteq H_3(\text{GL}_2)$, where the cup product is induced by the diagonal inclusion $\text{inc} : R^* \times \text{GL}_1 \rightarrow \text{GL}_2$. Moreover $\ker(\text{inc}_*)$ is a 2-torsion group.*

Proof. The only part that remains to be proved is that $\ker(\text{inc}_*)$ is a 2-torsion group. Let $x \in \ker(\text{inc}_*)$. For simplicity we may assume that $x = l_{a,b,c} = \mathbf{c}(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(c, c^{-1}))$, such that $a \otimes \{b, c\} + b \otimes \{a, c\} = 0$. Let Φ be the following composition

$$R^* \otimes K_2^M(R) \xrightarrow{\text{id}_{R^*} \otimes \iota} R^* \otimes H_2(\text{GL}_2) \xrightarrow{\cup} H_3(R^* \times \text{GL}_2) \xrightarrow{\alpha_*} H_3(\text{GL}_2),$$

where $\iota : K_2^M(R) \rightarrow H_2(\text{GL}_2)$ is described in the previous section, \cup is the cup product and $\alpha : R^* \times \text{GL}_2 \rightarrow \text{GL}_2$ is given by $(a, A) \mapsto aA$. It is easy to see that

$$\Phi(a \otimes \{b, c\}) = \mathbf{c}(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1})).$$

Now with easy computations, one sees that

$$\begin{aligned} 0 &= \Phi(0) \\ &= \Phi(a \otimes \{b, c\} + b \otimes \{a, c\}) \\ &= \mathbf{c}(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1})) \\ &\quad + \mathbf{c}(\text{diag}(b, b), \text{diag}(a, 1), \text{diag}(c, c^{-1})) \\ &= -2l_{a,b,c}. \end{aligned}$$

This completes the proof of the theorem. \square

Remark 3.2. One can show directly that if $a \otimes \{b, c\} + b \otimes \{a, c\} = 0$, then $l_{a,b,c} \in \ker(\text{inc}_* : H_3(\text{GL}_2) \rightarrow H_3(\text{GL}_3))$. To see this, let Ψ be the following composition

$$R^* \otimes K_2^M(R) \xrightarrow{\text{id}_{R^*} \otimes \iota} R^* \otimes H_2(\text{GL}_2) \xrightarrow{\cup} H_3(R^* \times \text{GL}_2) \longrightarrow H_3(\text{GL}_3).$$

Then it is easy to see that

$$\Psi(a \otimes \{b, c\}) = \mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, c, c^{-1})).$$

Now we have

$$\begin{aligned}
\text{inc}_*(l_{a,b,c}) &= +\mathbf{c}(\text{diag}(1, a, 1), \text{diag}(1, 1, b), \text{diag}(1, c, c^{-1})) \\
&= +\mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(c, c^{-1}, 1)) \\
&= -\mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, c, 1)) \\
&\quad -\mathbf{c}(\text{diag}(b, 1, 1), \text{diag}(1, a, 1), \text{diag}(1, c, 1)) \\
&= -\mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, 1, c)) \\
&\quad -\mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, c, c^{-1})) \\
&\quad -\mathbf{c}(\text{diag}(b, 1, 1), \text{diag}(1, a, 1), \text{diag}(1, 1, c)) \\
&\quad -\mathbf{c}(\text{diag}(b, 1, 1), \text{diag}(1, a, 1), \text{diag}(1, c, c^{-1})) \\
&= -\Psi(a \otimes \{b, c\} + b \otimes \{a, c\}) \\
&= 0.
\end{aligned}$$

Corollary 3.3. *Let R be a ring with many units.*

(i) *The natural map $\text{inc}_* : H_3(\text{GL}_2, \mathbb{Z}[1/2]) \rightarrow H_3(\text{GL}_3, \mathbb{Z}[1/2])$ is injective.*

(ii) *If $R^* = R^{*2} = \{a^2 | a \in R^*\}$, then $\text{inc}_* : H_3(\text{GL}_2) \rightarrow H_3(\text{GL}_3)$ is injective.*

Proof. The part (i) immediately follows from Theorem 3.1. Let $R^* = R^{*2}$. By Theorem 3.1, we may assume that $x \in \ker(\text{inc}_*)$ is of the form $l_{a,b,c} \in H_3(\text{GL}_2)$ such that $a \otimes \{b, c\} + b \otimes \{a, c\} = 0$. Let $c = c'^2$ for some $c' \in R^*$. Then $l_{a,b,c} = 2l_{a,b,c'}$ and $2(a \otimes \{b, c'\} + b \otimes \{a, c'\}) = 0$. Since $K_2^M(R)$ is uniquely 2-divisible [1, Proposition 1.2], $R^* \otimes K_2^M(R)$ is uniquely 2-divisible too. Hence $a \otimes \{b, c'\} + b \otimes \{a, c'\} = 0$. Now from Theorem 3.1, it follows that $2l_{a,b,c'} = 0$. Therefore $l_{a,b,c} = 0$ and hence $\text{inc}_* : H_3(\text{GL}_2) \rightarrow H_3(\text{GL}_3)$ is injective. \square

Example 3.4. Let $R = \mathbb{R}$. It is well-known that $K_2^M(\mathbb{R}) \simeq \langle \{-1, -1\} \rangle \oplus V$, where V is uniquely divisible and is generated by elements $\{a, b\}$ with $a, b > 0$. Let $l_{a,b,c} \in H_3(\text{GL}_2(\mathbb{R}))$ such that $a \otimes \{b, c\} + b \otimes \{a, c\} = 0$. If $a > 0$, then $a \otimes \{b, c\} = a \otimes \{-b, c\} = a \otimes \{b, -c\} = a \otimes \{-b, -c\}$, so we may assume that $b, c > 0$. Now with an argument as in the proof of the previous corollary, one sees that $l_{a,b,c} = 0$. A similar argument works if $b > 0$ or if $c > 0$. If $a, b, c < 0$, then one can easily reduce the problem to the case that $a = b = c = -1$, and it is trivial to see that $l_{-1,-1,-1} = 0$. Therefore $\text{inc}_* : H_3(\text{GL}_2(\mathbb{R})) \rightarrow H_3(\text{GL}_3(\mathbb{R}))$ is injective.

Remark 3.5. Consider the following chain of maps

$$R^{*\otimes 3} \otimes K_0^M(R) \xrightarrow{\delta_0^{(3)}} R^{*\otimes 2} \otimes K_1^M(R) \xrightarrow{\delta_1^{(3)}} R^* \otimes K_2^M(R) \xrightarrow{\delta_2^{(3)}} K_3^M(R) \rightarrow 0,$$

where

$$\begin{aligned}\delta_2^{(3)} : a \otimes \{b, c\} &\mapsto \{a, b, c\} \\ \delta_1^{(3)} : a \otimes b \otimes \{c\} &\mapsto a \otimes \{b, c\} + b \otimes \{a, c\} \\ \delta_0^{(3)} : a \otimes b \otimes c &\mapsto b \otimes c \otimes \{a\} + a \otimes c \otimes \{b\} + a \otimes b \otimes \{c\}.\end{aligned}$$

It is easy to see that this is, in fact, a chain complex. It is not difficult to see that $\ker(\delta_2^{(3)}) = \text{im}(\delta_1^{(3)})$ (see the proof of Theorem 3.2 in [5]). Under the composition

$$R^{*\otimes 3} \longrightarrow R^* \otimes H_2(R^*) \longrightarrow H_3(\text{GL}_2),$$

defined by

$$a \otimes b \otimes c \mapsto a \otimes \mathbf{c}(b, c) \mapsto \mathbf{c}(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(1, c)),$$

one can see that $\text{im}(\delta_0^{(3)})$ maps to zero. Thus we obtain a surjective map

$$\begin{aligned}\ker(\delta_1^{(3)})/\text{im}(\delta_0^{(3)}) &\longrightarrow \ker(H_3(\text{GL}_2) \rightarrow H_3(\text{GL}_3)), \\ \sum a \otimes b \otimes c + \text{im}(\delta_0^{(3)}) &\mapsto \sum l_{a,b,c}.\end{aligned}$$

Lemma 3.6. *Let R be a ring with many units.*

(i) *We have the exact sequence*

$$0 \longrightarrow H_3(\text{SL}_2, \mathbb{Z}[1/2])_{R^*} \longrightarrow H_3(\text{SL}, \mathbb{Z}[1/2]) \longrightarrow K_3^M(R)_{\mathbb{Z}[1/2]} \longrightarrow 0.$$

(ii) *If $R^* = R^{*2} = \{a^2 | a \in R^*\}$, then we have the exact sequence*

$$0 \longrightarrow H_3(\text{SL}_2) \longrightarrow H_3(\text{SL}) \longrightarrow K_3^M(R) \longrightarrow 0.$$

Proof. The proof is similar to the proof of Theorem 6.1 and Corollary 6.2 in [8]. \square

Theorem 3.7. *Let R be a ring with many units.*

(i) *We have the isomorphism*

$$K_3(R)^{\text{ind}} \otimes \mathbb{Z}[1/2] \simeq H_3(\text{SL}_2, \mathbb{Z}[1/2])_{R^*}.$$

(ii) *If $R^* = R^{*2} = \{a^2 | a \in R^*\}$, then*

$$K_3(R)^{\text{ind}} \simeq H_3(\text{SL}_2).$$

Proof. The proof is similar to the proof of Theorem 6.4 in [8]. \square

Remark 3.8. Previously Lemma 3.6 and Theorem 3.7 were only known for infinite fields [8, Corollary 6.2, Proposition 6.4].

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